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Folk Theorems in Multicriteria Repeated N-Person Games

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Abstract

In this paper we consider the question of existence of a multicriteria-Nash equilibrium in multicriteria multistage N-person games. Besides, we present several forms of multicriteria-Nash equilibrium for repeated games both with infinitely and finitely many stages.

Key Words: Repeated games, multicriteria games.

AMS subject classification: 91A20, 90B50, 91A35.

1 Introduction

The dynamic aspect of certain conflict situations may be missed when games are considered in pure strategic form. These features appear very often when games are played repeatedly and strategies must take into account the past behaviour of the players. In addition, in many real-world situations the agents must control more than a unique payoff function; Bergstresser and Yu (1977), Corley (1985), Fernández and Puerto (1996). This combination leads to the consideration of multicriteria multistage games. In the following, we consider the class of the multicriteria multistage non-zero-sum games, the existence of a multicriteria-Nash equilibrium and we prove extensions of folk theorems for this class of games. We are unaware of any results on existence of equilibria for this class of games. In addition, although there

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are many references to folk theorems in the literature of Game Theory, see e.g. Abreu et al. (1994), Friedman (1989), Gossner (1995) and the references therein, we are not aware of any of them dealing with multicriteria multistage games. This gap motivates our study in that these results are important because they describe how to reach rational behavior in this class of multistage games.

In this paper, we restrict ourselves to the simplest case where one or different games are played at a denumerable succession of points in time either finitely or infinitely many times. We prove the existence of multicriteria Nash equilibria for this class of multistage games and provide an easy class of equilibria characterized for some kind of behavior that prescribes loyalty to a tacit agreement as long as it is observed by everybody, but triggers retaliation as soon as someone breaks the agreement. This type of results is what we called multicriteria folk theorems. We present folk theorems for both finite and infinite number of stages multicriteria multistage games

The paper is organized as follows. The first section is devoted to introduce the class of games that we are to consider in the paper and to summarize the results in the paper. Section 2 formally introduces the model and proves that there always exists a multicriteria multistage Nash equilibrium in the considered class of games. In the third section we give sufficient conditions for a family of strategy profiles, the so called punishment strategies, to be multicriteria equilibria in the finite case; and a folk theorem for multicriteria repeated games with an infinite number of stages.

2 The model

We consider a sequence $(\Gamma^l)_{1\leq l\leq L}$ $(0 < 1 \leq L \leq +\infty)$ of multicriteria games that are played at a succession of given points in time. Each one of these games is an N-person game $\Gamma^l = (N, X_1^l, \ldots, X_n^l, K_1^l, \ldots, K_n^l)$, where N is the set of players, X_i^l is the set of pure strategies of player i and $K_i^l = (K_{i1}^l, \ldots, K_{im}^l)$ is the vector payoff function of player i, $i = 1 \ldots n$; $l = 1 \ldots L$. The game Γ^l is referred to as the l-th stage game. The process where all the single stage games are played successively is called multi-stage game (also supergame). We assume that our games have closed-loop perfect state information structure. Thus, players at any stage know the states of the game at previous stages. A strategy in the multi-stage game is an action plan that assigns a strategy in Γ^l in the l-th time epoch according to the information structure of the game. Notice that according to our assumption these strategies are the so called closed-loop strategies.

Definition 2.1. An L-stage N-person game G is a $2L+1$-tuple $G = (N, \mathcal{X}_1, \ldots, \mathcal{X}_n, H_1, \ldots, H_n)$ where $\mathcal{X}_i = \prod_{l=1}^{L} X_i^l$, $i = 1 \ldots n$ is the set of strategies of player i and for any $\mathrm{x} = (\mathrm{x}_1, \ldots, \mathrm{x}_n) \in \mathcal{X}_1 \times \ldots \times \mathcal{X}_n$, $H_i(\mathrm{x}) = \sum_{l=1}^{L} K_i^l(x_1^l, \ldots, x_n^l)$, $i = 1, \ldots, n$ is the vector payoff function of player i.

Note that for the ease of readability, we denote by $\mathrm{x}_i = (x_i^1, \ldots, x_i^L) \in \mathcal{X}_i$ the strategies of the i-th player, $i = 1 \ldots, n$. Then, we can introduce the concept of set of strategies in equilibrium (multicriteria Nash equilibrium).

Definition 2.2. A multi-criteria Nash equilibrium $\bar{\mathrm{x}}$ for the L-stage multicriteria game G is a strategy profile $\bar{\mathrm{x}} = (\bar{\mathrm{x}}_1, \ldots, \bar{\mathrm{x}}_n)$ where $\bar{\mathrm{x}}_i \in \mathcal{X}_i$ $i = 1, \ldots, n$ such that:

$$\bar{\mathrm{x}}_i \text{ is a weak-Pareto solution of } \mathrm{v}-\max_{\mathrm{x}_i \in \mathcal{X}_i} H_i(\bar{\mathrm{x}}_{-i}, \mathrm{x}_i), \quad \forall\, i = 1, \ldots, n. \tag{2.1}$$

Recall that $v - \max$ stands for the vector maximum problem; we say that x is a weak-Pareto solution of $v - \max_x f(x)$ (equivalently $x \in WE(f)$) if there does not exist another feasible solution y satisfying $f(y) > f(x)$, where $>$ means greater componentwise. Throughout this paper we will denote by $v - \max_x f(x)$ the set of weakly non-dominated values of this vector-maximum problem, that is $\mathrm{v} - \max f(x) := \{f(x) : x \in WE(f)\}$.

Definition 2.2 is not new and can be found in previous papers in the literature as for instance in Wang (1993), Borm et al. (1999) and the references therein. It is worth noting that it extends the rationale behind the classical definition of Nash equilibrium, Nash (1950), and that it also reduces to it when the number m of objective functions equals 1.

The first question the readers may put is whether any sequence of single stage multi-criteria Nash equilibria will result in a multi-criteria $L-stage$ Nash equilibrium. The answer is negative. This increases the interest in studying this class of games.

Example 2.1. Consider the two-stage repeated bicriteria game with payoff matrices: $K = (A_1, A_2)$, where

$$A_1 = \begin{bmatrix} (4,1) & (6,0) \\ (0,6) & (1,4) \end{bmatrix}, \quad A_2 = \begin{bmatrix} (1,4) & (0,6) \\ (6,0) & (4,1) \end{bmatrix}.$$

Let us denote by I_i (II_j) the $i-th$ pure strategy of player 1 (j-th pure strategy of player 2). The single stage game with payoff function given by K has 4 multi-criteria Nash equilibria: (I_1, II_1), (I_1, II_2), (I_2, II_1), (I_2, II_2). If we allow only pure strategies in the two-stage game, we have that the set of strategies for players I and II are:

Player I	$I_1 I_1$	$I_1 I_2$	$I_2 I_1$	$I_2 I_2$
Player II	$II_1 II_1$	$II_1 II_2$	$II_2 II_1$	$II_2 II_2$

These individual strategies give rise to 16 combinations that are the pure strategies of the supergame. Since payoffs will be the sum of payoffs in each period we have the following payoff matrices:

$$\bar{A}_1 = \begin{bmatrix} (8,2) & (10,1) & (10,1) & (12,0) \\ (4,7) & (5,5) & (6,6) & (7,4) \\ (4,7) & (6,6) & (5,5) & (7,4) \\ (0,12) & (1,10) & (1,10) & (2,8) \end{bmatrix}$$

$$\bar{A}_2 = \begin{bmatrix} (2,8) & (1,10) & (1,10) & (0,12) \\ (7,4) & (5,5) & (6,6) & (4,7) \\ (4,7) & (6,6) & (5,5) & (4,7) \\ (12,0) & (10,1) & (10,1) & (8,12)) \end{bmatrix}.$$

Take the following combination of Nash equilibria of the single stage game:

stage 1	stage 2
(I_1, II_1)	(I_2, II_2)

In the 2-stage game the strategy $\Big((I_1, II_1), (I_2, II_2)\Big)$ has a payoff for player I: $(5,5)$. It is clear that the strategy $\Big((I_2, II_1), (I_1, II_2)\Big)$ has a better payoff for player I that is $(6,6)$. Therefore, the strategy $\Big((I_1, II_1), (I_2, II_2)\Big)$ is not a Nash equilibrium in the 2-stage repeated game.

In spite of that example, in the case of repeated games it can be easily shown that the set of Nash equilibria is not empty because the repetition of the same Nash equilibrium in each single stage game is a Nash equilibrium in the supergame.

First of all, we must explain this assertion. We have defined the multi-criteria multi-stage Nash equilibrium $\mathrm{x} = (\mathrm{x}_1, \ldots, \mathrm{x}_n)$ as a n-tuple of L-tuples since $\mathrm{x}_i = (x_i^1, \ldots, x_i^L)$ for $i = 1, \ldots, n$. Nevertheless, we may see the equilibrium x in a different way rearranging its components. Just consider $x^l = (x_1^l, \ldots, x_n^l)$ for any $l = 1, \ldots, L$. Then we can also write $\mathrm{x} = (x^1, \ldots, x^L)$. After that transformation, x^l is a combination of strategies for the players in the game Γ^l. In this sense, we can also see x as a sequence of strategies of the players in the single stage games. Thus, it makes sense to study whether there are multi-stage Nash equilibrium whose components are Nash equilibria in the single-stage games.

Our next result states that the set of multi-criteria multi-stage Nash-equilibria is not empty and moreover that there always exists a multi-criteria multi-stage Nash-equilibrium which consists of multi-criteria Nash-equilibria of single-stage games.

For the ease of readability let us denote by $A \oplus B = \{c : c = a + b, a \in A, b \in B\}$.

Theorem 2.1. *For any multicriteria multi-stage game (not necessarily repeated) satisfying that the set of strategies X_i^l $i = 1, \ldots, n$ and $l = 1, \ldots, L$ is convex and compact and the vector payoff functions K_i^l are continuous and concave in x_i^l for any x_{-i}^l fixed, $i = 1, \ldots, n$ and $l = 1, \ldots, L$; there exist $\bar{x}^l = (\bar{x}_1^l, \ldots, \bar{x}_n^l)$ $l = 1, \ldots, L$ multicriteria Nash-equilibria in the l-th-stage game Γ^l $l = 1, \ldots, L$ such that the strategy profile $\bar{\mathrm{x}} = (\bar{\mathrm{x}}_1, \ldots, \bar{\mathrm{x}}_n)$ where $\bar{\mathrm{x}}_i = (\bar{x}_i^1, \ldots, \bar{x}_i^L)$ $i = 1, \ldots, n$ is a multi-stage Nash-equilibria in G.*

Proof. Because of the continuity and concavity of the payoff functions and the compactness and convexity of the strategy sets, the set of multicriteria Nash-equilibrium points is not empty (see Corollary 3.2 in Wang (1993)). Nash-equilibria of G are those strategy profiles $(\bar{\mathrm{x}}_1, \ldots, \bar{\mathrm{x}}_n)$ that hold:

$$\bar{\mathrm{x}}_i \text{ is a weak-Pareto solution of } \mathrm{v-}\max_{\mathrm{x}_i \in \mathcal{X}_i} H_i(\bar{\mathrm{x}}_{-i}, \mathrm{x}_i), \quad \forall\, i = 1, \ldots, n.$$

Now, $H_i(\bar{\mathrm{x}}_{-i}, \mathrm{x}_i) = \sum_{l=1}^L K_i^l(\bar{x}_{-i}^l, x_i^l)$. Then, since the sum function is stage-wise separable and monotonic we can apply the following recursive equations: (Notice that these equations are similar to Bellmann's principle but applied for multi-criteria dynamic programming as shown for instance in Mitten (1974) and Villarreal and Karwan (1982))

$$\begin{aligned} \mathrm{v} - \max\nolimits_{\mathrm{x}_i \in \mathcal{X}_i} H_i(\bar{\mathrm{x}}_{-i}, \mathrm{x}_i) = \quad & \mathrm{v} - \max\nolimits_{x_i^1 \in X_i^1, \ldots, x_i^{L-1} \in X_i^{L-1}} \Big(\sum\nolimits_{l=1}^{L-1} K_i^l(\bar{x}_{-i}^l, x_i^l) \\ & \oplus \mathrm{v-} \max_{x_i^L \in X_i^L} K_i^L(\bar{x}_{-i}^L, x_i^L) \Big), \qquad \forall\, i = 1, \ldots, n \end{aligned}$$

From this expression, any weak-Pareto solution $(\bar{x}_i^1, \dots, \bar{x}_i^L)$ of the lefthand problem, for any $i = 1, \dots, n$, admits a decomposition where the L component $\bar{x}_i^L$ is a weak-Pareto solution of the problem at the L-th stage problem. Therefore, at the L-stage it must exist $\bar{x}^L$ being a Nash-equilibria in Γ^L. Then, reasoning by backward induction we obtain the result. □

3 Repeated games

In this section we consider repeated rather than general multi-stage games. Note that under this hypothesis the single stage games Γ^l are all the same and therefore, we will omit the superscript in the notation.

For any strategy x_{-i} of set of players $N \setminus i$ in the game Γ, we denote by $v_{ij}(x_{-i}) = \max_{x_i} K_{ij}(x_{-i}, x_i)$, i.e. the best payoff that player i gets in his j-th criterion $j = 1, \dots, m$ against x_{-i}. Next, we introduce the set of security level vectors for the set of players $N \setminus i$:

$$V_i = \text{v-}\min_{x_{-i}} \Big(v_{i1}(x_{-i}), \dots, v_{im}(x_{-i}) \Big).$$

It is worth noting that V_i is the set of weakly non-dominated values of the above vector-minimum problem.

We also introduce the set

$$Eff_i = \{x_{-i} : (v_{i1}(x_{-i}), \dots, v_{im}(x_{-i})) \in V_i\}.$$

Every element of Eff_i will be denoted by $\tilde{x}_{-i}$. The component of $\tilde{x}_{-i}$ that corresponds to the strategy of player $j \neq i$ will be referred to as $(\tilde{x}_{-i})_j$.

Assume that we fix a Nash equilibrium $\bar{x}$ in Γ and denote by $\bar{\lambda}_i = K_i(\bar{x})$. Further assume that there exists a strategy $\hat{x}$ with $\hat{\lambda}_i = K_i(\hat{x})$ such that there exists at least one j with $\hat{\lambda}_j \underset{\neq}{>} \bar{\lambda}_j$. Moreover, assume that there does not exist $v_i \in V_i$ such that $v_i \geq \bar{\lambda}_i$ for all $i = 1, \dots, n$.

Consider the following family of strategies that we call punishment strategies:

Player i must play his strategy $\hat{x}_i$ on each stage l $(1 \leq l \leq s)$
if none of the players $j \neq i$ deviates from $\hat{x}_j$ on the stages

$k = 1, 2, \ldots, s-1$ and then he plays $\bar{x}_i$ on the remaining $L - s$ stages.

Denote by l^0 the first stage when a player, say j, first deviates from his $\hat{x}$ strategy (in other words l^0 is the number of the stage when the player i first time observes deviation of some player $j \neq i$ from his $\hat{x}_j$ strategy and $l^0 < s$), then player i must play $(\tilde{x}_{-j})_i$ for any $\tilde{x}_{-j} \in Eff_j$, from the stage $l^0 + 1$ until the end of the game at stage L.

It is clear that this family of strategies is correctly defined for all histories of the game. A similar family of strategies was already introduced although for single criterion games in Petrosjan and Egorova (2000). We can prove the following result.

Theorem 3.1. *Let G be a repeated game with a finite number L of stages. Then the n-tuple of punishment strategies is a multicriteria Nash equilibrium of G if and only if*

$$\nexists u_i \in U_i \textit{ such that } u_i > \hat{\lambda}_i + (L-s)\bar{\lambda}_i, \quad \forall\, i = 1, \ldots, n, \tag{3.1}$$

where $U_i = v\text{-}\max_{x_i} K_i(\hat{x}_{-i}, x_i) \oplus (L-s)V_i$.

Proof. Assume that everybody plays the punishment strategy. Then player j gets the payoff $\hat{\lambda}_j s + (L-s)\bar{\lambda}_j$. If player j abides by the punishment strategy up until period $s-1$, but in period s he deviates, then he can not get more than the payoffs in the following set:

$$\hat{\lambda}_j(s-1) \oplus \mathrm{v} - \max_{x_j} K_j(\hat{x}_{-j}, x_j) \oplus (L-s)V_j.$$

Thus, the condition for the n-tuple of punishment strategies $(\hat{x}, \bar{x})$ to be a multicriteria Nash equilibrium is that

$$\nexists u_i \in U_i \text{ such that } \hat{\lambda}_i(s-1) + u_i > \hat{\lambda}_i s + (L-s)\bar{\lambda}_i \quad \forall\, i = 1, \ldots, n.$$

Hence, the condition (3.1) holds. □

Let us denote by $\mathcal{K}_i = \mathrm{v} - \max_{x_i} K_i(\hat{x}_{-i}, x_i)$ for all $i = 1, \ldots, n$. In addition, we write, according to the previous theorem, the elements $u_i \in U_i$ as $u_i = \kappa_i + (L-s)v_i$, where $\kappa_i \in \mathcal{K}_i$ and $v_i \in V_i$. Our next result gives a constructive sufficient condition on the value s for the n-tuple of punishment strategies to be a multicriteria Nash equilibrium.

Theorem 3.2. *Let G be a repeated game with a finite number L of stages. If*

$$\min_{\kappa_j+(L-s)v_j\in U_j}\ \max_{\{k:(\bar\lambda_j-v_j)_k>0\}} \frac{(\hat\lambda_j-\kappa_j)_k+L(\bar\lambda_j-v_j)_k}{(\bar\lambda_j-v_j)_k}\geq s,$$

or

$$\max_{\kappa_j+(L-s)v_j\in U_j}\ \min_{\{k:(\bar\lambda_j-v_j)_k<0\}} \frac{(\hat\lambda_j-\kappa_j)_k+L(\bar\lambda_j-v_j)_k}{(\bar\lambda_j-v_j)_k}\leq s$$

$\forall j=1,\ldots,n$, then the n-tuple of punishment strategies is a multicriteria Nash equilibrium in G.

Proof. From condition (3.1) the n-tuple of punishment strategies is a Nash equilibrium iff for all $j=1,\ldots,n$ and all $u=\kappa_j+(L-s)v_j\in U_j$ there exists k, $1\leq k\leq m$ (notice that such index $k=k(j,u)$ because depends on j and u) such that

$$(\hat\lambda_j+(L-s)\bar\lambda_j)_k\geq u_k \quad \text{for some } k=1,\ldots,m.$$

Since $u_j=\kappa_j+(L-s)v_j$ this condition is equivalent to:

$$(\hat\lambda_j-\kappa_j)_k+L(\bar\lambda_j-v_j)_k\geq s(\bar\lambda_j-v_j)_k \quad \text{for some } k=1,\ldots,m.$$

Therefore, a sufficient condition is given by replacing, depending on the sign of $(\bar\lambda_j-v_j)_k$, the lefthand (righthand) term in the above inequality by a lower (upper) bound, i.e.

$$\min_{\kappa_j+(L-s)v_j\in U_j}\ \max_{\{k:(\bar\lambda_j-v_j)_k>0\}} \frac{(\hat\lambda_j-\kappa_j)_k+L(\bar\lambda_j-v_j)_k}{(\bar\lambda_j-v_j)_k}\geq s,$$

or

$$\max_{\kappa_j+(L-s)v_j\in U_j}\ \min_{\{k:(\bar\lambda_j-v_j)_k<0\}} \frac{(\hat\lambda_j-\kappa_j)_k+L(\bar\lambda_j-v_j)_k}{(\bar\lambda_j-v_j)_k}\leq s,$$

for any $j=1,\ldots,n$.

□

Obviously, the existence of such an n-tuple of punishment strategies is not trivial and it does not always exist. It will depend on the multicriteria-Nash equilibrium $\bar x$ that we choose and on the strategy $\hat x$. In fact, as we will show in the next examples there are choices so that the punishment strategy is a multicriteria-Nash equilibrium for some values of s and there are other choices where we cannot obtain multicriteria-Nash equilibrium by punishment strategies for any value of s.

Example 3.1. Consider the L repeated bicriteria game with payoff matrices: $K = (A_1, A_2)$, where

$$A_1 = \begin{bmatrix} (-1,1) & (1,1/2) \\ (-1,0) & (0,0) \end{bmatrix}, \quad A_2 = \begin{bmatrix} (-1,1) & (-3,1) \\ (-1,1) & (0,0) \end{bmatrix}.$$

In this game it is easily seen that the set of multicriteria-Nash equilibrium is given by:

$$\{(1,0)\} \times \{(q, 1-q) : q \in [0,1]\} \cup \{(p, 1-p) : p \in [0,1]\} \times \{(1,0)\},$$

where the first vector in the cartesian product are the strategies for player I, and the second vector the strategies for player II. In addition, p denotes the probability of playing the first strategy for the first player; and q is the probability that the second player plays his first strategy.

Let us choose $\bar{x} = ((0,1),(1,0))$ and $\hat{x} = ((1,0),(1,0))$. Thus $\bar{\lambda} = ((-1,0),(-1,1))$ and $\hat{\lambda} = ((-1,1),(-1,1))$. It is clear that $\hat{\lambda}_1 \underset{\neq}{\geq} \bar{\lambda}_1$.

The payoff functions in the game are:

$$K_1(p,q) = (-q + p(1-q), pq + p(1-q)/2)$$
$$K_2(p,q) = (-q - 3p(1-q), q(1-p) + p).$$

The security level set V_1 and V_2 are:

$$V_1 = \{(1-2q, (1+q)/2) : q \in [0,1]\}$$
$$V_2 = \{(-3p, 1) : 0 \leq p \leq 1/3\} \cup \{(-1,1) : 1/3 \leq p \leq 1\}.$$

Besides, the set $\mathcal{K}_i$ $i = 1,2$ are in this example:

$$\mathcal{K}_1 = \{(-1,p) : p \in [0,1]\},$$
$$\mathcal{K}_2 = \{(-3+2q, 1) : q \in [0,1]\}.$$

Applying Theorem 3.1 the n-tuple of punishment strategies is a multicriteria-Nash equilibrium iff there does not exist p, q satisfying all the conditions

(3.1). In our example one of the conditions is that there does exist an element of $\mathcal{K}_2 \oplus (L-s)V_2$ being greater than $\hat{\lambda}_2 + (L-s)\bar{\lambda}_2 = (-(L-s)-1, L-s+1)$. In this example:

$$\begin{aligned}\mathcal{K}_2 \oplus (L-s)V_2 = &\{(-3+2q-3p(L-s), 1+(L-s)) : 0 \le p \le 1/3, q \ge 0\} \\ &\cup\{(-3+2q-(L-s), 1+(L-s)) : 1/3 \le p \le 1,\ q \ge 0\}.\end{aligned}$$

However, for no $0 \le p \le 1$ the second component of the elements of $\mathcal{K}_2 \oplus (L-s)V_2$, namely $1+(L-s)$, can be greater than $1+(L-s)$, the second component of $\hat{\lambda}_2+(L-s)\bar{\lambda}_2$. Therefore, condition (3.1) is always fulfilled for any $1 \le s \le L$. Hence, the punishment strategy is always a multicriteria-Nash equilibrium.

Example 3.2. Consider the L repeated bicriteria game with payoff matrices: $K = (A_1, A_2)$, where

$$A_1 = \begin{bmatrix} (-1,1) & (1,1) \\ (-1,3) & (0,0) \end{bmatrix}, \quad A_2 = \begin{bmatrix} (-1,1) & (-3,1) \\ (-1,1) & (0,0) \end{bmatrix}.$$

In this game it is easily seen that the set of multicriteria-Nash equilibrium is given by:

$$\begin{gathered}\{(p,1-p) : 0 \le p \le 1/3\} \times \{(q,1-q) : 1/3 \le q \le 1\} \cup \\ \{(p,1-p) : 1/3 \le p \le 1\} \times \{(1,0)\} \cup \{(1,0)\} \times \{(q,1-q) : 0 \le q \le 1\},\end{gathered}$$

where p is the probability for playing the first strategy for the first player; and q is the probability that the second player plays his first strategy.

Let us choose $\bar{x} = ((1/3,2/3),(1/3,2/3))$ and $\hat{x} = ((1/3,2/3),(1,0))$. Thus $\bar{\lambda} = ((-1/9,1),(-1,5/9))$ and $\hat{\lambda} = ((-1,7/3),(-1,1))$. It is clear that $\hat{\lambda}_2 \underset{\neq}{>} \bar{\lambda}_2$.

The payoff functions in the game are:

$$\begin{aligned}K_1(p,q) &= (-q+p(1-q), p(1-3q)+3q) \\ K_2(p,q) &= (-q-3p(1-q), p+q(1-p)).\end{aligned}$$

The security level set V_1 and V_2 are:

$$\begin{aligned}V_1 &= \{(1-2q,1) : 0 \le q \le 1/3\} \cup \{(1-2q,3q) : 1/3 \le q \le 1\}, \\ V_2 &= \{(-3p,1) : 0 \le p \le 1/3\} \cup \{(-1,1) : 1/3 \le p \le 1\}.\end{aligned}$$

Besides, the set $\mathcal{K}_i$ and U_i $i = 1, 2$ are in this example:

$$\mathcal{K}_1 = \{(-1, 3-2p) : p \in [0,1]\},$$
$$\mathcal{K}_2 = \{(-1, 1/3 + 2/3q) : q \in [0,1]\}.$$

$$\begin{aligned} U_1 =& \{(-1 + (1-2q)(L-s), 3 - 2p + (L-s)) : 0 \leq p \leq 1,\ 0 \leq q \leq 1/3\} \\ & \cup \{(-1 + (1-2q)(L-s), 3 - 2p + 3q(L-s)) : 0 \leq p \leq 1,\ 1/3 \leq q \leq 1\}, \\ U_2 =& \{(-1 - 3p(L-s), 1/3 + 2/3q + (L-s)) : 0 \leq p \leq 1/3\} \\ & \cup \{(-1 - (L-s), 1/3 + 2/3q + (L-s)) : 1/3 \leq p \leq 1\}. \end{aligned}$$

In this example there are elements satisfying the conditions (3.1). It suffices to take $p < 1/3$ and $q \geq 1/3$. Therefore, since there are elements satisfying all the inequalities this n-tuple of punishment strategies is not a multicriteria-Nash equilibrium.

In order to obtain folk theorems for repeated games with an infinite number of stages we need to impose some more conditions. First, we need a discount factor so that the infinite summation is convergent. Let $0 < \alpha_i < 1$ be the discount factor of player i, $i = 1, \ldots, n$. Second, let us assume that there does not exist $v_i > \bar{\lambda}_i$, i.e. for all $v_i \in V_i$ there exists a $j(i)$ such that $\bar{\lambda}_{j(i)} \geq (v_i)_{j(i)}$.

We call trigger strategy, the strategy that for player i consists to play:

> If none of other players previously deviate from $\hat{x}$ strategy play $\hat{x}_i$. If $l-1$ is the first stage when one of the other players first deviate and that player is j then play $(\tilde{x}_{-j})_i$.

Theorem 3.3. *Let G be a repeated game with an infinite number of stages and discount factor $\alpha = (\alpha_i)_{i=1\ldots n}$. The n-tuple of trigger strategies is a Nash equilibrium if for any $i = 1, \ldots, n$*

$$\begin{aligned} & \max_{\{k:(\kappa_i - v_i)_k < 0\}} \frac{(\kappa_i - \hat{\lambda}_i)_k}{(\kappa_i - v_i)_k} \geq \alpha_i, \\ \textit{or} & \\ & \min_{\{k:(\kappa_i - v_i)_k > 0\}} \frac{(\kappa_i - \hat{\lambda}_i)_k}{(\kappa_i - v_i)_k} \leq \alpha_i \quad \forall \kappa_i \in \mathcal{K}_i,\ v_i \in V_i. \end{aligned} \tag{3.2}$$

Proof. Let us assume that every player plays without deviation the trigger strategy . Then, player i gets the vector payoff: $\frac{\hat{\lambda}_i}{1-\alpha_i}$. If player i abides by the strategy up until period $l-1$ and then he deviates then he can not get more than the payoffs in the set:

$$(1+\ldots+\alpha_i^{l-1})\hat{\lambda}_i \oplus \alpha_i \mathcal{K}_i \oplus V_i \sum_{r=l+1}^{\infty} \alpha_i^r,$$

where $\mathcal{K}_i = \mathrm{v} - \max_{x_i} K_i(\hat{x}_{-i}, x_i)$. The condition to be a Nash equilibrium in G is that for any $\kappa_i \in \mathcal{K}_i$ and $v_i \in V_i$ there exists a k (note that this k depends on κ_i and v_i) such that:

$$\frac{(\hat{\lambda}_i)_k}{1-\alpha_i} \geq \frac{1-\alpha_i^l}{1-\alpha_i}(\hat{\lambda}_i)_k + \alpha_i^l(\kappa_i)_k + \frac{\alpha_i^{l+1}}{1-\alpha_i}(v_i)_k.$$

This is equivalent to that for any $i = 1, \ldots, n$:

$$\alpha_i(\kappa_i - v_i)_k \geq (\kappa_i - \hat{\lambda}_i)_k \quad \forall \kappa_i \in \mathcal{K}_i,\ v_i \in V_i \text{ and some } k.$$

Hence, a sufficient condition is that for any $i = 1, \ldots, n$:

$$\max_{\{k:(\kappa_i - v_i)_k < 0\}} \frac{(\kappa_i - \hat{\lambda}_i)_k}{(\kappa_i - v_i)_k} \geq \alpha_i,$$

or

$$\min_{\{k:(\kappa_i - v_i)_k > 0\}} \frac{(\kappa_i - \hat{\lambda}_i)_k}{(\kappa_i - v_i)_k} \leq \alpha_i \quad \forall \kappa_i \in \mathcal{K}_i,\ v_i \in V_i.$$

□

Corollary 3.1. *In the same hypothesis that theorem 3.3, an n-tuple of trigger strategies is a Nash equilibrium in G if for any $i = 1, \ldots, n$:*

$$\min_{\substack{\kappa_i \in \mathcal{K}_i \\ v_i \in V_i}} \max_{\{k:(\kappa_i - v_i)_k < 0\}} \frac{(\kappa_i - \hat{\lambda}_i)_k}{(\kappa_i - v_i)_k} \geq \alpha_i,$$

or (3.3)

$$\max_{\substack{\kappa_i \in \mathcal{K}_i \\ v_i \in V_i}} \min_{\{k:(\kappa_i - v_i)_k > 0\}} \frac{(\kappa_i - \hat{\lambda}_i)_k}{(\kappa_i - v_i)_k} \leq \alpha_i.$$

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